Loss-calibrated Monte Carlo Action Selection

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Abstract

Bayesian decision-theory underpins robust decision-making in applications ranging from plant control to robotics where hedging action selection against state uncertainty is critical for minimizing low probability but potentially catastrophic outcomes (e.g, uncontrollable plant conditions or robots falling into stairwells). Unfortunately, belief state distributions in such settings are often complex and/or high dimensional, thus prohibiting the efficient application of analytical techniques for expected utility computation when real-time control is required. This leaves Monte Carlo evaluation as one of the few viable (and hence frequently used) techniques for online action selection. However, loss-insensitive Monte Carlo methods may require large numbers of samples to identify optimal actions with high certainty since they may sample from high probability regions that do not disambiguate action utilities. In this paper we remedy this problem by deriving an optimal proposal distribution for a loss-calibrated Monte Carlo importance sampler that bounds the regret of using an estimated optimal action. Empirically, we show that using our loss-calibrated Monte Carlo method yields *high-accuracy* optimal action selections in a *fraction* of the number of samples required by conventional loss-insensitive samplers.

Introduction

Bayesian decision-theory (Gelman et al. 1995; Robert 2001; Berger 2010) provides a formalization of robust decision-making in uncertain settings by maximizing expected utility. Formally, a utility function $\mathfrak{u}(\boldsymbol{\theta}, a)$ quantifies the return of performing an action $a \in \mathcal{A} = \{a_1, \ldots, a_k\}$ in a given state $\boldsymbol{\theta}$. When the true state is uncertain and only a belief state distribution $p(\boldsymbol{\theta})$ is known, Bayesian decision-theory posits that an optimal control action a should maximize the *expected utility* (EU)

$$\mathcal{U}(a) = \mathbb{E}[\mathfrak{u}(\boldsymbol{\theta}, a)] = \int \mathfrak{u}(\boldsymbol{\theta}, a) p(\boldsymbol{\theta}) d\boldsymbol{\theta}. \quad (1)$$

where by definition, the *optimal action* a^* is

$$a^* = \arg\max_a \quad \mathcal{U}(a). \quad (2)$$

In real-world settings such as robotics (Thrun 2000), the belief distribution $p(\boldsymbol{\theta})$ may be complex (e.g., highly multimodal) and/or high-dimensional thus prohibiting the application of analytical methods to evaluate the EU integral of (1). Practitioners often resort to the use of Monte Carlo methods to compute an approximate (but unbiased) expectation using n samples from $p(\boldsymbol{\theta})$:

$$\frac{1}{n} \sum_{i=1}^{n} [\mathfrak{u}(\boldsymbol{\theta}_i, a)], \quad \boldsymbol{\theta}_i \sim p. \quad (3)$$

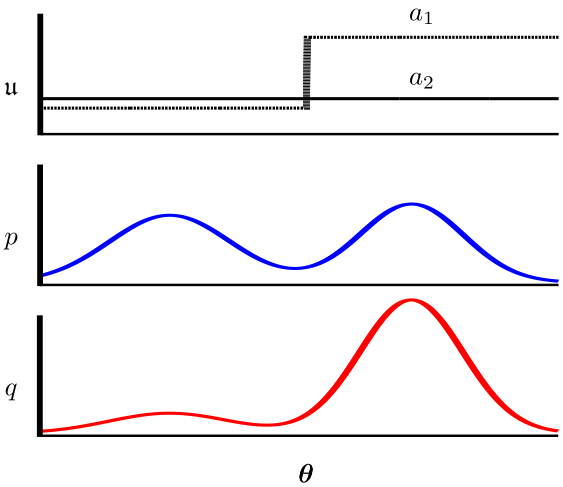


Figure 1: Motivation for loss-calibration in Monte Carlo action selection. (Top) Utility $\mathfrak{u}(\boldsymbol{\theta})$ for actions a_1 and a_2 as a function of state $\boldsymbol{\theta}$. (Middle) A belief state distribution $p(\theta)$ for which the optimal action $\arg\max_{a \in \{a_1, a_2\}} \mathbb{E}_p[\mathfrak{u}(\boldsymbol{\theta}, a)]$ should be computed. (Bottom) A potential proposal distribution $q(\boldsymbol{\theta})$ for importance sampling to determine the optimal action to take in $p(\boldsymbol{\theta})$.

Unfortunately, naïve application of Monte Carlo methods for optimal action selection often proves to be inefficient as we illustrate in Figure 1. At the top we show two utility functions for actions a_1 and a_2 as a function of univariate state θ on the x-axis. Below this, in blue, we show the known state belief distribution $p(\theta)$. Here, it turns out that $\mathcal{U}(a_1) > \mathcal{U}(a_2)$. Unfortunately, if we sample from $p(\theta)$ to compute a Monte Carlo expected utility for each of a_1 and a_2, we find ourselves sampling frequently in the region where $\mathfrak{u}(\boldsymbol{\theta}, a_1)$ and $\mathfrak{u}(\boldsymbol{\theta}, a_2)$ are very close, but where sub-optimal a_2 is marginally better than a_1.

An intuitive remedy to this problem is provided by an importance sampling approach (see e.g. (Geweke 1989)) where we sample more heavily in regions as indicated by distribution $q(\theta)$ and then reweight the Monte Carlo expectation to provide an unbiased estimate of $\mathcal{U}(a)$. Formally, the theory of importance sampling tells us that since

$$\mathcal{U}(a) = \int \left[\frac{\mathrm{u}(\boldsymbol{\theta}, a)p(\boldsymbol{\theta})}{q(\boldsymbol{\theta})}\right] q(\boldsymbol{\theta})d\boldsymbol{\theta}, \tag{4}$$

we can draw samples from q to compute an (unbiased) estimate of $\mathcal{U}(a)$ as

$$\hat{\mathcal{U}}_n(a) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\mathrm{u}(\boldsymbol{\theta}_i, a)p(\boldsymbol{\theta}_i)}{q(\boldsymbol{\theta}_i)}\right], \quad \boldsymbol{\theta}_i \sim q. \tag{5}$$

This leaves us with one key question to answer in this paper: *How can we automatically derive a $q(\boldsymbol{\theta})$ to increase the probability that the optimal action a^* is selected for a finite set of n samples?* Answering this question is important for real-time applications of Bayesian decision-theoretic approaches where efficiency and optimality are two key operating criteria.

To this end, in the subsequent section we derive an optimal proposal distribution for a loss-calibrated Monte Carlo importance sampler. To do this, we first show connections between regret and the probability of non-optimal action selection and then further connect an upper bound on the latter to the variance of our expected utility estimator. We are then able to derive an optimal proposal distribution q that minimizes this variance through the calculus of variations.

We evaluate our loss-calibrated Monte Carlo method in two domains. We first examine synthetic plant control examples building on those of (Lacoste-Julien, Huszar, and Ghahramani 2011), who were also motivated by loss-calibration in Bayesian decision theory, albeit not in the case of Monte Carlo methods as we focus on in this work. We also demonstrate results in a Bayesian decision-theoretic robotics setting with uncertain localization motivated by the work of (Thrun 2000).

In both empirical settings and in states with up to 100 dimensions, we demonstrate that using our loss-calibrated Monte Carlo method yields high-accuracy optimal action selections in a fraction of the number of samples required by conventional loss-insensitive samplers to achieve the same level of accuracy. This suggests a new class of loss-calibrated Monte Carlo samplers for efficient online Bayesian decision-theoretic action selection.

Loss-calibrated Monte Carlo Importance Sampling

In many applications of decision theory, sampling is the most time-consuming step. Since we know these samples are ultimately used to estimate high-utility actions, we are interested in guiding the sampler to be more efficient for this task.

Here, we will pick a distribution q to draw samples from, which are in turn used to select the action that maximizes the EU. The estimated optimal action $\hat{a}_n$ from n samples is defined as

$$\hat{a}_n = \arg\max_a \quad \hat{\mathcal{U}}_n(a). \tag{6}$$

Since the samples are drawn randomly from q, $\hat{a}_n$ is a random variable and so is its expected utility $\mathcal{U}(\hat{a}_n)$. As such, we use $\mathbb{E}, \mathbb{P}$ and $\mathbb{V}$ henceforth to denote the expectation, probability and variance operators. We emphasize that all random variables are determined by q.

In principle, we would like to select the distribution q to minimize regret, i.e. maximize the true EU of the estimated action $\hat{a}_n$. As this is challenging to do directly, we proceed in three steps:

1. We establish a connection between regret and the probability of non-optimal action selection in Theorem 1.
2. Since calculating the probability of selecting the non-optimal action is intractable to be directly minimized, we derive an upper bound in Theorem 3, based on the variance of the difference of estimated utilities.
3. Theorem 5 shows how to calculate the distribution q to minimize this bound.

Minimizing regret

To find the optimal estimated action $\hat{a}_n$ with fewer samples, we wish to select q that minimizes the regret. Formally we define this as

$$\min_q \quad \ell(\hat{a}_n) \quad = \quad \mathbb{E}\left[\mathcal{U}(a^*) - \mathcal{U}(\hat{a}_n)\right]. \tag{7}$$

Direct minimization of Equation 7 is difficult, hence we bound it with the probability of selecting a non-optimal action instead. Tightening this bound with respect to q will lead to a practical strategy. It is detailed in the following theorem.

Theorem 1 (Regret bounds). *For the optimal action a^* and its estimate $\hat{a}_n$ the regret as defined in Equation 7, is bounded as*

$$\Delta\, \mathbb{P}\left[a^* \neq \hat{a}_n\right] \leq \ell(\hat{a}_n) \leq \Gamma\, \mathbb{P}\left[a^* \neq \hat{a}_n\right], \tag{8}$$

where $\Delta = \mathcal{U}(a^*) - \max_{a' \in \mathcal{A}\setminus\{a^*\}} \mathcal{U}(a')$ and $\Gamma = \mathcal{U}(a^*) - \min_{a' \in \mathcal{A}} \mathcal{U}(a')$.

Proof. We know $\mathbb{E}\left[\mathcal{U}(\hat{a}_n)\right]$ is equal to

$$\begin{aligned}
&\mathbb{P}\left[a^* = \hat{a}_n\right]\mathcal{U}(a^*) + \sum_{a \in \mathcal{A}\setminus\{a^*\}} \mathbb{P}\left[a = \hat{a}_n\right]\mathcal{U}(a) \\
\geq &\mathbb{P}\left[a^* = \hat{a}_n\right]\mathcal{U}(a^*) + \sum_{a \in \mathcal{A}\setminus\{a^*\}} \mathbb{P}\left[a = \hat{a}_n\right] \min_{a' \in \mathcal{A}} \mathcal{U}(a') \\
= &\mathbb{P}\left[a^* = \hat{a}_n\right]\mathcal{U}(a^*) + \mathbb{P}\left[a^* \neq \hat{a}_n\right] \min_{a' \in \mathcal{A}} \mathcal{U}(a').
\end{aligned}$$

This is equivalent to stating that $\ell(\hat{a}_n) \leq \Gamma\, \mathbb{P}\left[a^* \neq \hat{a}_n\right]$ after some manipulation. Similarly, we have that

$$\mathbb{E}\left[\mathcal{U}(\hat{a}_n)\right] \leq \mathbb{P}\left[a^* = \hat{a}_n\right]\mathcal{U}(a^*) + \mathbb{P}\left[a^* \neq \hat{a}_n\right] \max_{a' \in \mathcal{A}\setminus\{a^*\}} \mathcal{U}(a')$$

which leads to $\ell(\hat{a}_n) \geq \Delta \mathbb{P}\left[a^* \neq \hat{a}_n\right]$. □

The bound is very intuitive: minimizing the probability of the estimated optimal action $\hat{a}_n$ being non-optimal will lead to a bound on the regret. Clearly, for two actions we have $\Delta = \Gamma$. Thus, in the two-action case, minimizing the probability of selecting a non-optimal action is *equivalent* to maximizing the expected utility of the selected action. With more actions, these objectives are not equivalent, but we can see that the difference is controlled in terms of Δ and Γ.

Minimizing the probability of non-optimal action

We now turn to the problem of minimizing $\mathbb{P}\left[a^* \neq \hat{a}_n\right]$. In the following, Lemma 2 provides a bound on the probability of non-optimal action. Further details and another view of the same problem with slightly better bounds is provided in supplementary material.

In the subsequent lemma, we upper bound the indicator function with a smooth and convex upper bound that will be easier to minimize. The use of *surrogate* function for minimizing indicator has also been used in similar problems (see e.g. (Bartlett, Jordan, and McAaliffe 2006)).

Lemma 2. *For an optimal action a^* and its estimate $\hat{a}_n$ obtained from sampling, we have $\forall t > 0$,*

$$\mathbb{P}\left[a^* \neq \hat{a}_n\right] \leq \sum_{a \neq a^*} \sum_{a' \neq a} \mathbb{E}\left[\left(t\left(\hat{\mathcal{U}}_n(a) - \hat{\mathcal{U}}_n(a')\right) + 1\right)^2\right].$$

Proof. Firstly, decompose $\mathbb{P}\left[a^* \neq \hat{a}_n\right]$ as

$$\begin{aligned}\sum_{a \neq a^*} \mathbb{P}[a = \hat{a}_n] &\leq \sum_{a \neq a^*} \sum_{a' \neq a} \mathbb{P}[\hat{\mathcal{U}}_n(a) > \hat{\mathcal{U}}_n(a')] \\ &= \sum_{a \neq a^*} \sum_{a' \neq a} \mathbb{E}\left[\mathbb{I}\left[\hat{\mathcal{U}}_n(a) > \hat{\mathcal{U}}_n(a')\right]\right].\end{aligned}$$

Applying the inequality that $\mathbb{I}[v > 0] \leq (tv + 1)^2$ gives the result. □

The next theorem then relates the value inside the above expectation to the variance, thus bonding the probability of incorrect action selection by the sum of variances.

Theorem 3 (Upper bound on the probability of non-optimal actions)**.** *We have the following upper bound on the probability of non-optimal action selection for k actions in set $\mathcal{A}$, true expected utility $\mathcal{U}(a)$ and its estimation $\hat{\mathcal{U}}_n(a)$ obtained from finite samples:*

$$\begin{aligned}\mathbb{P}\left[a^* \neq \hat{a}_n\right] \leq \sum_{a \neq a^*} \sum_{a' \neq a} \Big(&1 + 2t\Big(\mathcal{U}(a) - \mathcal{U}(a')\Big) \\ &+ t^2 \mathbb{V}\left[\hat{\mathcal{U}}_n(a) - \hat{\mathcal{U}}_n(a')\right] + t^2\Big(\mathcal{U}(a) - \mathcal{U}(a')\Big)^2\Big),\end{aligned} \tag{9}$$

Proof. Expand the quadratic in Lemma 2 and use $\mathbb{E}[X^2] = \mathbb{V}[X] + \mathbb{E}[X]^2$ and $\mathbb{E}[\hat{\mathcal{U}}_n(a) - \hat{\mathcal{U}}_n(a')] = \mathcal{U}(a) - \mathcal{U}(a')$. □

In general, one would like to select the constant t to minimize this bound. As this depends on the variance, which is a function of q and the number of samples n, this is difficult to do analytically. However, we expect the variance to decrease as n increases, so we instead derive a value for t that leads to an asymptotically tight upper bound.

Lemma 4. *For the bounds detailed in Theorem 3, if the variance term is zero, the value of t that minimizes the upper bound is*

$$t = \frac{\sum_{a \neq a^*} \sum_{a' \neq a} \mathcal{U}(a') - \mathcal{U}(a)}{\sum_{a \neq a^*} \sum_{a' \neq a} \left(\mathcal{U}(a) - \mathcal{U}(a')\right)^2}.$$

Proof. In general, the value of t minimizing $2ta + t^2 b$ is $t = -a/b$. □

The critical feature of Equation 9 is that all terms on the RHS other than the variance are constant with respect to the sampling distribution q. Thus, this theorem suggests that a reasonable surrogate to minimize the regret in Equation 7 and consequently maximize the expected utility of the estimated optimal action is to minimize the variance of the difference of the estimated utilities. This result is quite intuitive — if we have a low-variance estimate of the differences of utilities, we will tend to select the best action.

This is aligned with the importance sampling literature where it is well known that the optimal distribution to sample from is the one that minimizes the variance (Rubinstein 1981; Glasserman 2004). Our analysis shows the variance of the function that has to be minimized is of a particular form that depends on the difference of the utilities (rather than each utility independently).

Optimal q

We established that to find the optimal proposal distribution q^* (i.e. optimal q), we minimize the sum of variances obtained from Theorem 3. Since a^* is unknown, we sum over all actions in $\mathcal{A}$, rather than just $\mathcal{A}\backslash\{a^*\}$. Since everything except variance in Equation 9 is independent of q, we formulate the objective

$$\min_q \sum_{a \in \mathcal{A}} \sum_{a' \in \mathcal{A}\backslash\{a\}} \mathbb{V}\left[\hat{\mathcal{U}}_n(a) - \hat{\mathcal{U}}_n(a')\right] \text{ s.t.} \int q(\boldsymbol{\theta}) d\boldsymbol{\theta} = 1. \tag{10}$$

Here, the constraint on q is to ensure the resulting solution is a proper probability distribution.

The following theorem provides the solution to the optimization problem in Equation 10 that we are interested in.

Theorem 5. *Let $\mathcal{A} = \{a_1, \ldots, a_k\}$ with non-negative utilities. The optimal distribution $q^*(\boldsymbol{\theta})$ is the solution to problem in Equation 10 and has the following form:*

$$q^*(\boldsymbol{\theta}) \propto p(\boldsymbol{\theta}) \sqrt{\sum_{a \in \mathcal{A}} \sum_{a' \in \mathcal{A}\backslash\{a\}} (\mathfrak{u}(\boldsymbol{\theta}, a) - \mathfrak{u}(\boldsymbol{\theta}, a'))^2}. \tag{11}$$

Proof. Since we know $\mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$, for computing the objective in Equation 10 the second expectation becomes $(\mathcal{U}(a) - \mathcal{U}(a'))^2$ and is independent of q, then we only need to minimize $\mathbb{E}\left[\left(\hat{\mathcal{U}}_n(a) - \hat{\mathcal{U}}_n(a')\right)^2\right]$. Consider this value for a particular pair (a, a'). Denoting $\Upsilon(\boldsymbol{\theta}_i, a, a') = \mathfrak{u}(\boldsymbol{\theta}, a) - \mathfrak{u}(\boldsymbol{\theta}, a')$, this is equal to

$$\begin{aligned}&\int q(\boldsymbol{\theta}_{1,\ldots,n}) \left(\frac{1}{n} \sum_{i=1}^{n} \frac{\Upsilon(\boldsymbol{\theta}_i, a, a') p(\boldsymbol{\theta}_i)}{q(\boldsymbol{\theta}_i)}\right)^2 d\boldsymbol{\theta}_{1,\ldots,n} \\ &= \frac{1}{n^2} \int \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\Upsilon(\boldsymbol{\theta}_i, a, a') \Upsilon(\boldsymbol{\theta}_j, a, a') p(\boldsymbol{\theta}_i) p(\boldsymbol{\theta}_j)}{q(\boldsymbol{\theta}_i) q(\boldsymbol{\theta}_j)} \\ &\qquad \times q(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_n) d\boldsymbol{\theta}_{1,\ldots,n}.\end{aligned}$$

Since all the samples are independent, $q(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_n) = q(\boldsymbol{\theta}_1) \ldots q(\boldsymbol{\theta}_n)$. Now if $i \neq j$, it is easy to see that q vanishes and those terms become independent of q. If $i = j$ however, only one of the terms in the denominator cancels out with the joint. Also because the sum is over similar terms, we have n times the same expression, leading to the Lagrangian of

$$\frac{1}{n}\sum_{a\in\mathcal{A}}\sum_{a'\in\mathcal{A}\setminus\{a\}}\int\frac{\Upsilon(\boldsymbol{\theta},a,a')^2p(\boldsymbol{\theta})^2}{q(\boldsymbol{\theta})}d\boldsymbol{\theta}+\lambda\left(\int q(\boldsymbol{\theta})d\boldsymbol{\theta}-1\right).$$

Taking the derivative with respect to $q(\boldsymbol{\theta})$, we have that

$$-\frac{1}{n}\sum_{a\in\mathcal{A}}\sum_{a'\in\mathcal{A}\setminus\{a\}}\frac{\Upsilon(\boldsymbol{\theta},a,a')^2p(\boldsymbol{\theta})^2}{q(\boldsymbol{\theta})^2}+\lambda=0$$

which concludes the theorem since λn only induces a proportionality constant. □

This is quite intuitive – the samples $\boldsymbol{\theta}$ will be concentrated on regions where $p(\boldsymbol{\theta})$ is large, and the difference of utilities between the actions is large, which is precisely the intuition that motivated our work in Figure 1. This will tend to lead to the empirically optimal action being the true one, i.e. that $\hat{a}_n$ approaches a^*.

In practice, the normalization constants for p and q are likely to be unknown, meaning that direct use of Eq. 5 is impossible. However, there are well-known self-normalized variants that can be used in practice with $p(\boldsymbol{\theta})\propto\tilde{p}(\boldsymbol{\theta})$ and $q(\boldsymbol{\theta})\propto\tilde{q}(\boldsymbol{\theta})$, namely

$$\hat{\mathcal{U}}_n(a)=\frac{1}{n}\sum_{i=1}^{n}\mathfrak{u}(\boldsymbol{\theta}_i,a)\frac{\tilde{p}(\boldsymbol{\theta}_i)}{\tilde{q}(\boldsymbol{\theta}_i)}\Bigg/\frac{1}{n}\sum_{i=1}^{n}\frac{\tilde{p}(\boldsymbol{\theta}_i)}{\tilde{q}(\boldsymbol{\theta}_i)}\quad\boldsymbol{\theta}_i\sim\tilde{q}. \tag{12}$$

This simply means that for the case of unnormalized $\tilde{p}$ and $\tilde{q}$ and letting the $\frac{1}{n}$ terms cancel, the summed utility values have to now be reweighted by the slightly more complex value of $\left(\frac{\tilde{p}(\boldsymbol{\theta}_i)}{\tilde{q}(\boldsymbol{\theta}_i)}\Big/\sum_{j=1}^{n}\frac{\tilde{p}(\boldsymbol{\theta}_j)}{\tilde{q}(\boldsymbol{\theta}_j)}\right)$.

Furthermore, as it is hard to directly sample q, we must resort to Markov Chain Monte Carlo (MCMC) methods (Neal 1993), e.g. Metropolis-Hastings (MH). This disregards an important aspect, namely that the samples we obtain for q are not truly independent. Rather, the number of effective samples are affected by the mixing rate of the Markov chain. Our derivation above does not account for these mixing rates, which could be important in many applications. For this reason, our experiments will distinguish between two settings: First, one can run an extremely long chain, and subsample from this, approximating nearly independent samples as in the derivation above, which we call `Subsampled MC`. Second, one can run a single Markov chain, as would be typical in practice, which we call `Sequential MC`.

Applications

As discussed earlier, many applications require optimal actions to be selected efficiently given known (but complex) p and $\mathfrak{u}$. In this section we provide applications and evaluate how well the samples drawn from p and q^* compare. In these simulations we are interested in finding the optimal action, i.e., the one that maximizes the expected utility, with the minimum number of samples. As such we generate samples from the true distribution p and the proposed optimal distribution q^* (obtained from Theorem 5 as per the application's specifications) and compute the expected utilities for each action. In case direct sampling is not possible we use Metropolis-Hastings MCMC by initializing the chain at a random point and using a Normal distribution centered at the current sample with isotropic covariance optimally tuned so that around 23% of samples are accepted (Roberts, Gelman, and Gilks 1997). In each experiment n samples are generated 200 times and the mean of the percentage of times the true optimal action is selected is reported.

We include two diagnostics for MCMC samplers: in the first one (`Subsampled MC`) we have generated a large chain of 100000 samples and selected random subsamples to compute the best action using the empirical expected utilities $\hat{\mathcal{U}}_n(a)$. Since samples drawn from Markov chains are typically correlated, this diagnosis will help ensure samples are independent. In the second diagnostic (`Sequential MC`) we draw samples sequentially from a single Markov Chain started at a random point to calculate the expected utilities for selecting the best action.

Power-plant Control

We consider a power plant where the temperature of the generator has to be maintained in a safe range following the example from (Lacoste-Julien, Huszar, and Ghahramani 2011); the only actions available to achieve this are to turn the generator on or off. Suboptimal action choices that keep the generator on in high temperatures or turn it off in unnecessary cases when the temperature is safe and no maintenance is required lead to financial loss for the power plant and should be avoided.

For this problem, we can model the distribution of the temperature and use a high utility for cases where a safe action of turning the generator on or off is taken, formally,

$$\mathfrak{u}(\boldsymbol{\theta},a=\texttt{on}/\texttt{off})=\begin{cases}H_a & c_{a,1}^{(d)}<\boldsymbol{\theta}^{(d)}<c_{a,2}^{(d)}\\ L_a & \text{otherwise}\end{cases} \tag{13}$$

where $\boldsymbol{\theta}^{(d)}$ is the d-th dimension of the temperature, H_a, L_a (for action $a\in\texttt{on}/\texttt{off}$) is the reward for the given action a in the temperature intervals defined by $c_{a,1}^{(d)}$ and $c_{a,2}^{(d)}$. We use three distinct one dimensional utilities for simulations: (i) $c_{\texttt{on},1}^{(1)}=15, c_{\texttt{on},2}^{(1)}=20, c_{\texttt{off},1}^{(1)}=15, c_{\texttt{off},2}^{(1)}=21, H_{\texttt{on}}=6.5, H_{\texttt{off}}=5, L_{\texttt{on}}=1.5, L_{\texttt{off}}=4$; (ii) $c_{\texttt{on},1}^{(1)}=45, c_{\texttt{on},2}^{(1)}=50, c_{\texttt{off},1}^{(1)}=35, c_{\texttt{off},2}^{(1)}=40, H_{\texttt{on}}=6.5, H_{\texttt{off}}=3, L_{\texttt{on}}=1.5, L_{\texttt{off}}=2.5$; (iii) $c_{\texttt{on},1}^{(1)}=5, c_{\texttt{on},2}^{(1)}=+\infty, c_{\texttt{off},1}^{(1)}=-\infty, c_{\texttt{off},2}^{(1)}=+\infty, H_{\texttt{on}}=5, H_{\texttt{off}}=3, L_{\texttt{on}}=2.5, L_{\texttt{off}}=3$.

Corresponding to each utility, the following three distributions are used to demonstrate how the samples drawn from p and q^* obtained from Theorem 5 perform in selecting the optimal action:

(i) $p(\boldsymbol{\theta})=0.7*\mathcal{N}(\boldsymbol{\theta}^{(1)};3,7)+0.3*\mathcal{N}(\boldsymbol{\theta}^{(1)};12,2)$ where $\mathcal{N}(\boldsymbol{\theta}^{(1)};\mu,\sigma^2)$ is a normal distribution with mean μ and variance σ^2;

(ii) $p(\boldsymbol{\theta})=0.05*\mathcal{N}(\boldsymbol{\theta}^{(1)};3,1)+0.2*\mathcal{N}(\boldsymbol{\theta}^{(1)};6,1)+0.05*\mathcal{N}(\boldsymbol{\theta}^{(1)};10,3)+0.3*\mathcal{N}(\boldsymbol{\theta}^{(1)};15,2)+0.05*\mathcal{N}(\boldsymbol{\theta}^{(1)};20,7)+0.1*\mathcal{N}(\boldsymbol{\theta}^{(1)};25,2)+0.05*\mathcal{N}(\boldsymbol{\theta}^{(1)};30,3)+0.2*\mathcal{N}(\boldsymbol{\theta}^{(1)};40,5)$

(iii) a log-normal distribution $p(\boldsymbol{\theta})=\text{Log-}\mathcal{N}(\boldsymbol{\theta}^{(1)};0,1)$.

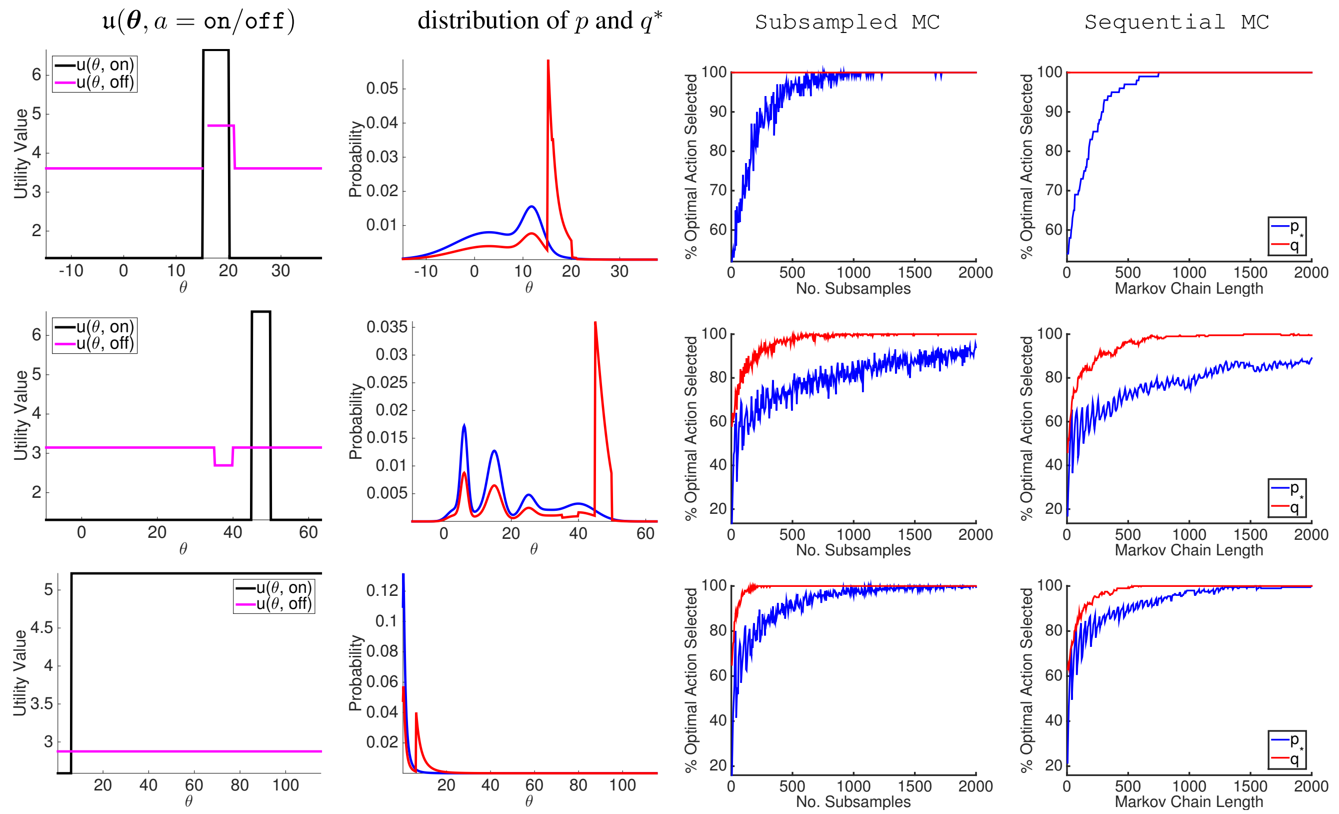
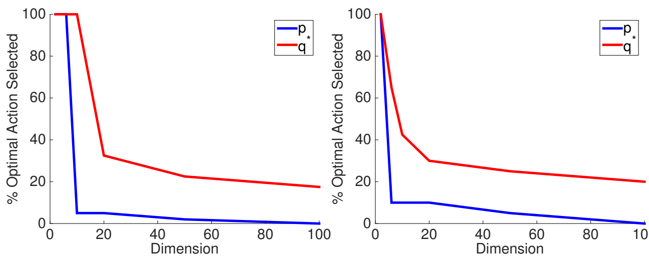


Figure 2: Power-plant simulations: the step-valued utility function (as in Equation 13) in the first column, the true distribution p (in blue) and q^* (in red) in the second column and in the third and forth columns the result of performing Subsampled MC and Sequential MC (as described in the text) are shown. In the two right-hand columns, note that q^* achieves the same percentage of optimal action selection performance as p in a mere fraction of the number of samples.

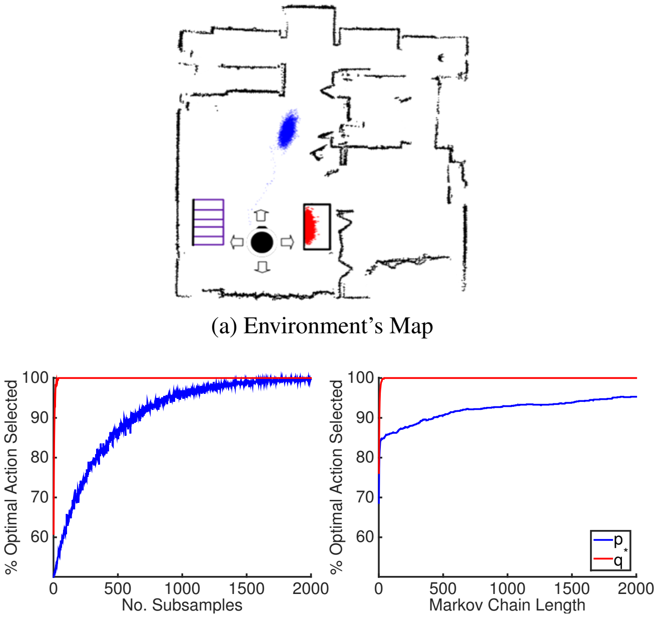
In Figure 2, the utility functions are shown in the first column (with black indicating the utility of on as detailed in Equation 13) and the temperature distributions (p in blue as described above and q^* in red) in the second column are shown. In the third and fourth columns the result of performing Subsampled MC and Sequential MC of the Metropolis-Hastings sampler for selecting the best action is shown such that the x-axis represents the number of samples and the y-axis shows the percentage of times the correct optimal action is selected. Here, in general, we observe that a significantly smaller number of samples from q^* is needed to select the best action in comparison to the number of samples from p required to achieve the same performance.

To investigate the performance of samples from p and q^* in higher dimensions, we use a d-dimensional Gaussian mixture corresponding to temperatures at each point in the plant as $p(\boldsymbol{\theta}) = \mathcal{N}(\boldsymbol{\theta}; \mathbf{10}, \boldsymbol{\Sigma}) + \mathcal{N}(\boldsymbol{\theta}; \mathbf{20}, \boldsymbol{\Gamma})$ where $\mathbf{10}$ and $\mathbf{20}$ are d-dimensional vectors with constant value 10 and 20 as the mean and $\boldsymbol{\Sigma}_{i,j} = 5 + \mathbb{I}[i = j]$ and $\boldsymbol{\Gamma}_{i,j} = 3 + 7\mathbb{I}[i = j]$ as $d \times d$ covariance matrix. In addition, the utility function in Equation 13 is specified with $c^{(d)}_{\mathtt{on},1} = 23, c^{(d)}_{\mathtt{on},2} = 25, c^{(d)}_{\mathtt{off},1} = 20, c^{(d)}_{\mathtt{off},2} = 22, H_{\mathtt{on}} = 50d, H_{\mathtt{off}} = 13, L_{\mathtt{on}} = 1.1, L_{\mathtt{off}} = 1.5\log(d)$. In Figure 3 for $d \in \{2, 4, 10, 20, 50, 80, 100\}$, we observe that in an average of 100 runs of the MCMC with 200 samples, as the dimensions increase using q^* is more efficient. In fact, for a 100-dimensional bimodal Gaussian we are unable to find the optimal action using only 200 samples from p, which should be contrasted with the significantly improved performance given by sampling from q^*.



(a) Subsampled MC (b) Sequential MC

Figure 3: Performance of the decision maker in selecting the best action as the dimension of the problem increases in the power-plant. Note that at 100 dimensions, p is unable to select the optimal action whereas q still manages to select it a fraction of the time (and would do better if more samples were taken).



(a) Environment's Map

(b) `Subsampled MC`

(c) `Sequential MC`

Figure 4: A robot's internal map showing the samples taken from its true belief distribution p (two modes are shown in blue, the second one is slightly obfuscated by the robot) and the optimal sampling distribution q^* derived by our loss calibrated Monte Carlo importance sampler in 4a. In 4b and 4c we see the performance (in terms of percentage of optimal action selected) of our loss-calibrated sampling method using q^* leads to near immediate detection of the optimal action in only a few samples.

Robotics

Another application where sampling has been commonly used is localization in robotics (Thrun 2000). It is a risk-sensitive-decision making problem where the robot is rewarded for navigating to the charger in order to maintain its power but wrong actions may lead to catastrophic incidents like falling down the stairs or crashing into obstacles. Due to minimal resources on-board a robot and the nature of the real-time localization problem, it is crucial for the robot to be able to select the optimal action rapidly, yet safely.

The state of the robot is the combination of its coordinates on a map and its heading direction. In our example for these experiments, we use a three dimensional Gaussian belief state distribution with two locations in a room intended to model that a robot's belief update has been confused by symmetries in the environment: one mode is at the robot's true location and the other at the opposite end of the room.

In this experiment, we consider a map as shown in Figure 4a where there is a flat in-door environment that the robot can move by selecting one of the four actions forward, backward, right or left. This action will lead to a movement step in robot from the current point on map with the heading direction towards the selected action. In doing so however, the robot has to avoid the stairs (low utility region) and select the charging source (high utility region).

Assuming a deterministic transition dynamics model $\boldsymbol{\theta}'\!=\!T(\boldsymbol{\theta},a)$ and denoting $(T(\boldsymbol{\theta}_x,a),T(\boldsymbol{\theta}_y,a))$ as the location of the robot after taking action a from state $\boldsymbol{\theta}$ (that is, moving from the current location in the direction of the the selected action heading) and $\mathcal{R}_{\mathtt{r}}$ the set induced by region $\mathtt{r}$, we use the following utility function:

$$\mathtt{u}(\boldsymbol{\theta},a)=\begin{cases} H & (T(\boldsymbol{\theta}_x,a),T(\boldsymbol{\theta}_y,a))\in\mathcal{R}_{\mathtt{charger}} \\ L & (T(\boldsymbol{\theta}_x,a),T(\boldsymbol{\theta}_y,a))\in\mathcal{R}_{\mathtt{stair}} \\ M & \text{otherwise} \end{cases}, \qquad (14)$$

where $L < M < H$ (in our experiments: $L = 1, M = 10, H = 400$) and $a \in \{\mathtt{forward}, \mathtt{backward}, \mathtt{right}, \mathtt{left}\}$. Using distribution q^* from Theorem 5 as illustrated in Figure 4a, the samples from q^* (in red) concentrated on the charger's location which has higher utility value compared to the samples from p (in blue) that are from the mode of the distribution.

As shown in Figure 4b and 4c, using distribution q^* and running the same diagnostics as the previous experiment we see significant improvement in selection of the optimal action, requiring only a fraction of the samples of p to achieve the same optimal action selection percentage.

Conclusion and Future Work

We investigated the problem of loss-calibrated Monte Carlo importance sampling methods to improve the efficiency of optimal Bayesian decision-theoretic action selection in comparison to conventional loss-insensitive Monte Carlo methods. We derived an optimal importance sampling distribution to minimize the regret bounds on the expected utility for multiple actions. This, to the best of our knowledge, is the first result linking the utility function for actions and the optimal distribution for Monte Carlo importance sampling in Bayesian decision theory. We drew connections from regret to the probability of selecting non-optimal actions and from there to the variance. We showed using an alternative distribution as derived in Theorem 5 will sample more heavily from regions of significance as identified by their sum of utility differences.

Empirically, we showed that our loss-calibrated Monte Carlo method yields high-accuracy optimal action selections in a fraction of the number of samples required by loss-insensitive samplers in synthetic examples of up to 100 dimensions and robotics-motivated applications.

Future work should investigate the extension of the novel results in this work to the case of (a) continuously parameterized actions (Alessandro, Restelli, and Bonarini 2007), (b) imprecise utility functions (e.g, when the return of a state is not known precisely, but can be sampled) (Boutilier 2003), (c) uncontrollable sampling (where the utility partially depends on auxiliary variables that cannot be directly sampled from) and (d) applications in active learning and crowdsourcing (Beygelzimer, Dasgupta, and Langford 2009). Furthermore, the bounds obtained here are not tight in the multi-action setting and can be improved in future work.

Altogether, this work and the many avenues of further research it enables suggest a new class of state-of-the-art loss-calibrated Monte Carlo samplers for efficient online Bayesian decision-theoretic action selection.

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