Symbolic Variable Elimination for Discrete and Continuous Graphical Models

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Abstract

Probabilistic reasoning in the real-world often requires inference in continuous variable graphical models, yet there are few methods for *exact, closed-form* inference when joint distributions are non-Gaussian. To address this inferential deficit, we introduce SVE – a *symbolic* extension of the well-known *variable elimination* algorithm to perform exact inference in an *expressive* class of mixed discrete and continuous variable graphical models whose conditional probability functions can be well-approximated as oblique piecewise polynomials with bounded support. Using this representation, we show that we can compute all of the SVE operations *exactly and in closed-form*, which crucially includes *definite integration w.r.t. multivariate piecewise polynomial functions*. To aid in the efficient computation and compact representation of this solution, we use an extended algebraic decision diagram (XADD) data structure that supports all SVE operations. We provide illustrative results for SVE on probabilistic inference queries inspired by robotics localization and tracking applications that mix various continuous distributions; this represents the first time a general closed-form exact solution has been proposed for this expressive class of discrete/continuous graphical models.

Introduction

Real-world probabilistic reasoning is rife with uncertainty over continuous random variables with complex (often nonlinear) relationships, e.g., estimating the position and pose of entities from measurements in robotics, or radar tracking applications with asymmetric stochastic dynamics and complex mixtures of noise processes. While closed-form exact solutions exist for inference in some continuous variable graphical models, such solutions are largely limited to relatively well-behaved cases such as joint Gaussian models. On the other hand, the more complex inference tasks of tracking with real-world sensors involves underlying distributions that are beyond the reach of current closed-form exact solutions. Take, for example, the robot sensor model in Figure 1 motivated by (Thrun et al. 2000). Here $d \in \mathbb{R}$ is a variable representing the distance of a mobile robot to a wall

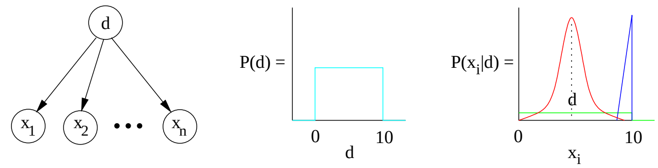


Figure 1: The *robot localization* graphical model and all conditional probabilities.

and x_i are the observed measurements of distance (e.g., using a laser range finder). The prior distribution $P(d)$ is uniform $U(d; 0, 10)$ while the observation model $P(x_i|d)$ is a *mixture* of three distributions: (red) is a truncated Gaussian representing noisy measurements x_i of the actual distance d within the sensor range of $[0, 10]$; (green) is a uniform noise model $U(d; 0, 10)$ representing random anomalies leading to any measurement; and (blue) is a triangular distribution peaking at the maximum measurement distance (e.g., caused by spurious deflections of laser light or the transparency of glass).

While our focus in this paper is on exact inference in *general* discrete and continuous variable graphical models and not purely on this robotics localization task, this example clearly motivates the real-world need for reasoning with complex distributions over continuous variables. However, in contrast to previous work that has typically resorted to (sequential) Monte Carlo methods (Thrun et al. 2000; Doucet, De Freitas, and Gordon 2001) to perform inference in this model (and dynamic extensions) via *sampling*, our point of departure in this work is to seek *exact, closed-form solutions* to general probabilistic inference tasks in these graphical models, e.g., arbitrary conditional probability queries or conditional expectation computations.

To achieve this task, we focus on an expressive class of mixed discrete and continuous variable Bayesian networks whose conditional probability functions can be expressed as oblique piecewise polynomials with bounded support (where oblique piece boundaries are represented by conjunctions of linear inequalities). In practice, this representation is sufficient to represent or arbitrarily approximate a wide range of common distributions such as the following: uniform (piecewise constant), triangular (piecewise linear), truncated normal distributions[1] (piecewise quadratic or

[1]In practice, measurements have natural ranges that make trun-

quartic), as well as all mixtures of such distributions as exemplified in $P(x_i|d)$ above (since a sum of piecewise polynomials is still piecewise polynomial). While polynomials directly support the integrals we will need to compute *variable elimination* (Zhang and Poole 1996) in closed-form, computing the definite integrals of polynomials w.r.t. arbitrary linear piece boundaries turns out to be a much more difficult task achieved through novel *symbolic* methods that underlie the key contribution of *symbolic variable elimination (SVE)* that we make in this paper.

In the following sections, we present our graphical model framework using a *case* representation of probability distributions followed by a description of our SVE procedure. After this, we introduce the *extended algebraic decision diagram (XADD)* — a practical data structure for efficiently and compactly manipulating the case representation — and apply XADD-based SVE to robotics localization and a tracking tasks demonstrating exact closed-form inference for these problems.

Discrete and Continuous Graphical Models

Factor Graph Representation

A discrete and continuous *graphical model* compactly represents a joint probability distribution $p(\mathbf{b}, \mathbf{x})$ over an assignment $(\mathbf{b}, \mathbf{x}) = (b_1, \ldots, b_n, x_1, \ldots, x_m)$ to respective random variables $(\mathbf{B}, \mathbf{X}) = (B_1, \ldots, B_n, X_1, \ldots, X_m)$.[2] Each b_i $(1 \leq i \leq n)$ is boolean s.t. $b_i \in \{0, 1\}$ and each x_j $(1 \leq j \leq m)$ is continuous s.t. $x_j \in \mathbb{R}$.

As a general representation of both directed and undirected graphical models, we use a *factor graph* (Kschischang, Frey, and Loeliger 2001) representing a joint probability $p(\mathbf{b}, \mathbf{x})$ as a product of a finite set of factors F, i.e.,

$$p(\mathbf{b}, \mathbf{x}) = \frac{1}{Z} \prod_{f \in F} \Psi_f(\mathbf{b}_f, \mathbf{x}_f). \tag{1}$$

Here, $\mathbf{b}_f$ and $\mathbf{x}_f$ denote the subset of variables that participate in factor f and $\Psi_f(\mathbf{b}_f, \mathbf{x}_f)$ is a non-negative, real-valued potential function that can be viewed as gauging the local compatibility of assignments $\mathbf{b}_f$ and $\mathbf{x}_f$. The functions Ψ_f may not necessarily represent probabilities and hence a normalization constant Z is often required to ensure $\sum_{\mathbf{b}} \int_{\mathbf{x}} p(\mathbf{b}, \mathbf{x}) d\mathbf{x} = 1$.

We will specify all of our algorithms on graphical models in terms of general factor graphs, but we note that *Bayesian networks* represent an important modeling formalism that we will use to specify our examples in this paper. For the Bayesian network directed graph in the introductory example, the joint distribution for n variables is represented as the product of all variables conditioned on their parent variables in the graph and can be easily converted to factor graph form, i.e.,

$$\begin{aligned} p(d, x_1, \ldots, x_k) &= p(d) \prod_{i=1}^{k} p(x_i|d) \\ &= \psi_d(d) \prod_{i=1}^{k} \Psi_{x_i}(x_i, d) \end{aligned} \tag{2}$$

where quite simply, $\Psi_d(d) := p(d)$ and $\Psi_{x_i}(x_i, d) := p(x_i|d)$. Here, $Z = 1$ and is hence omitted since joint distributions represented by Bayesian networks marginalize to 1 by definition.

Variable Elimination Inference

Given a joint probability distribution $p(\mathbf{b}, \mathbf{x})$ defined by a factor graph, our objective in this paper will be to perform two types of *exact, closed-form* inference:

(Conditional) Probability Queries: for query variables $\mathbf{q}$ and disjoint (possibly empty) evidence variables $\mathbf{e}$ drawn from a subset of $(\mathbf{b}, \mathbf{x})$, we wish to infer the *exact* form of $p(\mathbf{q}|\mathbf{e})$ as a *function* of $\mathbf{q}$ and $\mathbf{e}$. This can be achieved by the following computation, where for notational convenience we assume variables are renamed s.t. $\{\mathbf{b} \cup \mathbf{x}\} \setminus \{\mathbf{q} \cup \mathbf{e}\} = (b_1, \ldots, b_s, x_1, \ldots, x_t)$ and $\mathbf{q} = (b_{s+1}, \ldots, b_{s'}, x_{t+1}, \ldots, x_{t'})$:

$$p(\mathbf{q}|\mathbf{e}) = \tag{3}$$
$$= \frac{\sum_{b_1} \cdots \sum_{b_s} \int \cdots \int_{\mathbb{R}^t} \prod_{f \in F} \Psi_f(\mathbf{b}_f, \mathbf{x}_f)\, dx_1 \ldots dx_t}{\sum_{b_1} \cdots \sum_{b_{s'}} \int \cdots \int_{\mathbb{R}^{t'}} \prod_{f \in F} \Psi_f(\mathbf{b}_f, \mathbf{x}_f)\, dx_1 \ldots dx_{t'}}$$

The $\frac{1}{Z}$ from (1) would appear in *both* the numerator and denominator and hence cancels.

(Conditional) Expectation Queries: for a continuous query random variable Q and disjoint evidence assignment $\mathbf{e}$ drawn from a subset of $(\mathbf{b}, \mathbf{x})$, we wish to infer the *exact value* of $\mathbb{E}[Q|\mathbf{e}]$.[3] This can be achieved by the following computation, where $p(q|\mathbf{e})$ can be pre-computed according to (3):

$$\mathbb{E}[Q|\mathbf{e}] = \int_{q=-\infty}^{\infty} [q \cdot p(q|\mathbf{e})]\, dq \tag{4}$$

Variable elimination (VE) (Zhang and Poole 1996) is a simple and efficient algorithm given in Algorithm 1 that exploits the distributive law to *efficiently* compute each elimination step (i.e., any $\sum_{b_i}$ or $\int_{x_j}$ in (3) and (4)) by first factorizing out all factors independent of the elimination variable; this prevents unnecessary multiplication of factors, hence minimizing the size and complexity of each elimination operation. If the factor representation is closed and computable under the VE operations of multiplication and marginalization then VE can compute any (conditional) probability or expectation query in (3) and (4). Closed-form, exact solutions for VE are well-known for the discrete and joint Gaussian cases; next we extend VE to expressive piecewise polynomial discrete/continuous graphical models.

Symbolic Variable Elimination in Discrete/Continuous Graphical Models

As discussed previously, piecewise polynomial functions provide an expressive framework for representing dis-

cated variants of distributions appropriate, e.g., a range finder that exhibits Gaussian distributed measurement errors but only returns measurements in the range $[0, 10]$ may be well-suited to a truncated Normal distribution observation model.

[2] Notational comments: we sometimes abuse notation and treat vectors of random variables or assignments as a set, e.g., $(\mathbf{b}, \mathbf{x}) = \{b_1, \ldots, b_n, x_1, \ldots, x_m\}$. Also we often do not distinguish between a random variable (upper case) and its realization (lower case), e.g., $p(x_1) := p(X_1 = x_1)$.

[3] We do not discuss the expectation of $\{0, 1\}$ boolean random variables B since $\mathbb{E}[B|\mathbf{e}] = P(B = 1|e)$, which can already be computed in (3).

crete/continuous variable graphical models when all distributions have bounded support. In this section, we introduce a case notation and operations for piecewise polynomials, define factor graphs in terms of these case statements, and show that all VE operations, including definite integration w.r.t. multivariate oblique piecewise polynomials, can be computed in exact, closed-form using a purely *symbolic* representation, hence the algorithm *Symbolic VE* (SVE).

Case Representation and Operators

For this section, we will assume all functions are represented in *case* form as follows:

$$f = \begin{cases} \phi_1 & f_1 \\ \vdots & \vdots \\ \phi_k & f_k \end{cases} \tag{5}$$

Here, the f_i may be polynomials of $\mathbf{x}$ with non-negative exponents. The ϕ_i are logical formulae defined over $(\mathbf{b}, \mathbf{x})$ that can consist of arbitrary conjunctions of (a) (negated) boolean variables in $\mathbf{v}$ and (b) inequalities $(\geq, >, \leq, <)$, where the left and right operands must be *linear* functions (required to represent oblique piecewise boundaries). We assume that the set of conditions $\{\phi_1, \ldots, \phi_k\}$ disjointly and exhaustively partition $(\mathbf{b}, \mathbf{x})$ such that f is well-defined for all $(\mathbf{b}, \mathbf{x})$. It is easy to verify that such a representation can represent the uniform and triangular distributions and arbitrarily approximate the truncated normal distribution required to specify $p(x_i|d)$ discussed in the introduction.

Unary operations such as scalar multiplication $c \cdot f$ (for some constant $c \in \mathbb{R}$) or negation $-f$ on case statements f are straightforward; the unary operation is simply applied to each f_i ($1 \leq i \leq k$). Intuitively, to perform a *binary operation* on two case statements, we simply take the cross-product of the logical partitions of each case statement and perform the corresponding operation on the resulting paired partitions. Thus, we perform the "cross-sum" $\oplus$ of two (unnamed) cases as follows:

$$\begin{cases} \phi_1: & f_1 \\ \phi_2: & f_2 \end{cases} \oplus \begin{cases} \psi_1: & g_1 \\ \psi_2: & g_2 \end{cases} = \begin{cases} \phi_1 \wedge \psi_1: & f_1 + g_1 \\ \phi_1 \wedge \psi_2: & f_1 + g_2 \\ \phi_2 \wedge \psi_1: & f_2 + g_1 \\ \phi_2 \wedge \psi_2: & f_2 + g_2 \end{cases}$$

Likewise, we perform $\ominus$ and $\otimes$ by, respectively, subtracting or multiplying partition values (rather than adding them) to obtain the result. Some partitions resulting from the application of the $\oplus$, $\ominus$, and $\otimes$ operators may be inconsistent (infeasible); if we can detect this (e.g., via a linear constraint solver), we may simply discard such partitions as they are irrelevant to the function value.

For variable elimination, we'll need to compute definite integrals — a fairly non-trivial operation that is discussed in its own section. But first we discuss maximization (needed for working with integral bounds) and restriction (needed to compute marginals for boolean variables).

Symbolic maximization is fairly straightforward to define if we note that the conditional nature of the case statements allows us to directly encode maximization:

$$\max\left(\begin{cases} \phi_1: & f_1 \\ \phi_2: & f_2 \end{cases}, \begin{cases} \psi_1: & g_1 \\ \psi_2: & g_2 \end{cases} \right) = \begin{cases} \phi_1 \wedge \psi_1 \wedge f_1 > g_1: & f_1 \\ \phi_1 \wedge \psi_1 \wedge f_1 \leq g_1: & g_1 \\ \phi_1 \wedge \psi_2 \wedge f_1 > g_2: & f_1 \\ \phi_1 \wedge \psi_2 \wedge f_1 \leq g_2: & g_2 \\ \phi_2 \wedge \psi_1 \wedge f_2 > g_1: & f_2 \\ \phi_2 \wedge \psi_1 \wedge f_2 \leq g_1: & g_1 \\ \phi_2 \wedge \psi_2 \wedge f_2 > g_2: & f_2 \\ \phi_2 \wedge \psi_2 \wedge f_2 \leq g_2: & g_2 \end{cases}$$

The key observation here is that case statements are closed under the $\max$ operation (similarly for $\min$). While it may appear that this representation will lead to an unreasonable blowup in size, we note the XADD that we introduce later will be able to exploit the internal decision structure of this maximization to represent it much more compactly.

The two operations required for marginalization over boolean variables b are $\oplus$ and *restriction* of variable b in factor f to the value $x \in 0, 1$, written as $f|_b = x$. For $x = 1$ ($x = 0$), $f|_b = x$ simply requires instantiating all variables b in f with x. For example, let

$$f = \begin{cases} \phi_1 \wedge b: & f_1 \\ \phi_2 \wedge \neg b: & f_2 \\ \phi_3: & f_3 \end{cases}$$

then the two possible restrictions of b yield the following results (where inconsistent case partitions have been removed):

$$f|_{b=1} = \begin{cases} \phi_1: & f_1 \\ \phi_3: & f_3 \end{cases} \qquad f|_{b=0} = \begin{cases} \phi_2: & f_2 \\ \phi_3: & f_3 \end{cases}.$$

Algorithm 1: VE(F, *order*)

input : F, *order*: a set of factors F, and a variable *order* for elimination
output: a set of factors after eliminating each $v \in order$

begin
  // *eliminate each v in the given order*
  foreach $v \in order$ **do**
    // *multiply $\otimes$ all factors containing v*
    // *into f_v, put other factors in $F_{\setminus v}$*
    $f_v \leftarrow 1$; $F_{\setminus v} \leftarrow \emptyset$
    foreach $f \in F$ **do**
      if (f *contains* v)
      then $f_v \leftarrow f_v \otimes f$
      else $F_{\setminus v} \leftarrow F_{\setminus v} \cup \{f\}$
    // *eliminate var; insert result into factor*
    // *set F along with $F_{\setminus v}$*
    if (*var is boolean*)
    then $F \leftarrow F_{\setminus v} \cup \{\sum_{v \in \{0,1\}} f_v\}$
    else $F \leftarrow F_{\setminus v} \cup \{\int_{v=-\infty}^{\infty} f_v \, dv\}$
  return F
end

Definite Integration of the Case Representation

One of the major technical contributions of this paper is the symbolic computation of the definite integration required to eliminate continuous variables in SVE. If we are computing $\int_{x_1=-\infty}^{\infty} f \, dx_1$ for f in (5), we can rewrite it in the following equivalent form

$$\int_{x_1=-\infty}^{\infty} \sum_i \mathbb{I}[\phi_i] \cdot f_i \, dx_1 \;=\; \sum_i \int_{x_1=-\infty}^{\infty} \mathbb{I}[\phi_i] \cdot f_i \, dx_1 \quad (6)$$

where $\mathbb{I}[\cdot]$ is an indicator function taking value 1 when its argument is true, 0 when it is false. Hence we can compute the integrals separately for each case partition (producing a case statement) and then $\sum$ the results using $\oplus$.

To continue with the integral for a single case partition, we introduce a concrete example. Let $f_1 := x_1^2 - x_1 x_2$ and $\phi_1 := [x_1 > -1] \wedge [x_1 > x_2 - 1] \wedge [x_1 \le x_2] \wedge [x_1 \le x_3 + 1] \wedge [x_2 > 0]$. In computing $\int_{x_1=-\infty}^{\infty} \mathbb{I}[\phi_1] \cdot f_1 \, dx_1$, the first thing we note is that the *linear* constraints involving x_1 in $\mathbb{I}[\phi_1]$ can be used to restrict the integration range for x_1. From these constraints, we can see that the integrand can only be non-zero for $\max(x_2-1,-1) < x_1 \le \min(x_2, x_3+1)$. Using the $\max$ operation defined previously, we can write explicit functions in *piecewise polynomial case form* for these respective lower and upper bounds LB and UB:

$$LB := \begin{cases} x_2 - 1 > -1 : x_2 - 1 \\ x_2 - 1 \le -1 : -1 \end{cases} \quad UB := \begin{cases} x_2 < x_3 + 1 : x_2 \\ x_2 \ge x_3 + 1 : x_3 + 1 \end{cases}$$

Now we can rewrite the integral as[4]

$$\mathbb{I}[x_2 > 0] \int_{x_1=LB}^{UB} (x_1^2 - x_1 x_2) \, dx_1. \quad (7)$$

Note here that $\mathbb{I}[x_2 > 0]$ is independent of x_1 and hence can factor outside the integral. With all indicator functions moved into the LB or UB or factored out, we can now compute the integral:

$$\mathbb{I}[x_2 > 0] \left[\frac{1}{3} x_1^3 - \frac{1}{2} x_1^2 x_2 \Big|_{x_1=LB}^{x_1=UB} \right]. \quad (8)$$

The question now is simply how to do this evaluation? Here we note that every expression (variables, constants, indicator functions, etc.) can be written as a simple case statement or as operations on case statements, even the upper and lower bounds as shown previously. So the evaluation is simply

$$\mathbb{I}[x_2 > 0] \otimes \Bigg[\left(\frac{1}{3} UB \otimes UB \otimes UB \ominus \frac{1}{2} UB \otimes UB \otimes (x_2) \right) \ominus \left(\frac{1}{3} LB \otimes LB \otimes LB \ominus \frac{1}{2} LB \otimes LB \otimes (x_2) \right) \Bigg]. \quad (9)$$

Hence the result of the definite integration over a case partition of a piecewise polynomial function with linear constraints is simply a case statement in the same form — this is somewhat remarkable given that all of the bound computations were symbolic. Furthermore, one might fear that high-order operations like $UB \otimes UB \otimes UB$ could lead to a case partition explosion, but we note this example simply has the effect of cubing the expressions in each partition of UB since all case partitions are mutually disjoint.

However, we are not yet done, there is one final step that we must include for correctness. Because our bounds are symbolic, it may be the case for some assignment to $(\mathbf{b}, \mathbf{x})$ that $LB \ge UB$. In this case the integral should be zero since the constaints on x_1 could not be jointly satisfied. To enforce this symbolically, we simply need to $\otimes$ (9) by case statements representing the following inequalities for all pairs of upper and lower bounds:

$$\mathbb{I}[x_2 > x_2 - 1] \otimes \mathbb{I}[x_2 > -1] \otimes \mathbb{I}[x_3 + 1 > x_2 - 1] \otimes \mathbb{I}[x_3 + 1 > -1] \quad (10)$$

Of course, here $\mathbb{I}[x_2 + 1 > x_2 - 1]$ could be removed as a tautology, but the other constraints must remain.

This provides the solution for a single case partition and from (6), we just need to $\oplus$ the case statements resulting from each definite $\int$ evaluation to obtain the final result, *still in oblique piecewise polynomial case form.*

Piecewise Polynomial Factor Graphs and Symbolic Variable Elimination (SVE)

We define a factor graph from (1) with factors represented by our case formalism as *piecewise polynomial factor graphs (PPFGs)*. With the preceding machinery, generalizing VE to Symbolic VE (SVE) for PPFGs is straightforward. We need only replace all sums and products in VE with the case versions $\oplus$ and $\otimes$. Then the only operations left to specify in the VE Algorithm 1 are the computations for the variable eliminations. For PPFGs, there are two cases:

Discrete Variable Elimination (line 14 of VE):

$$\sum_{v \in \{0,1\}} f_v := f_v|_{v=0} \oplus f_v|_{v=1}.$$

Continuous Variable Elimination (line 15 of VE):

$$\int_{v=-\infty}^{\infty} f_v \, dv := \text{(see definite integration)}$$

Since definite integration and all other required SVE operations preserve the case property, all SVE operations can be computed on a PPFG in closed-form.[5]

This completes the definition of SVE. If the exact model could be represented as a PPFG, then SVE provides exact closed-form inference. Otherwise one can approximate most discrete and continuous graphical models to arbitrary precision using PPFGs — once this is done, SVE will provide exact inference in this approximated model.

Extended ADDs (XADDs) for Case Statements

In practice, it can be prohibitively expensive to maintain a case statement representation of a value function with explicit partitions. Motivated by algebraic decision diagrams (ADDs) (Bahar et al. 1993), which maintain compact representations for finite discrete functions, we use an extended

[4]The careful reader will note that because the lower bounds were defined in terms of $>$ rather than $\le$, we technically have an improper integral and need to take a limit. However, in taking the limit, we note that all integrands are continuous polynomials of order 0 or greater, so the limit exists and yields the same answer as substituting the limit value. Hence for polynomials, we need not be concerned about whether bounds are inclusive or not.

[5]While the $\max(\cdot,\cdot)$ and $\min(\cdot,\cdot)$ case operations (defined previously and required for lower and upper bound computation in definite integration) can theoretically introduce new nonlinear constraints $f_i \ge g_j$, we note that these $\max$ and $\min$ operations are only ever applied to linear bound expressions (derived from linear case constraints) during definite integration and hence only ever introduce new linear constraints. This is a crucial observation that ensures definite integration can always be applied in SVE for PPFGs.

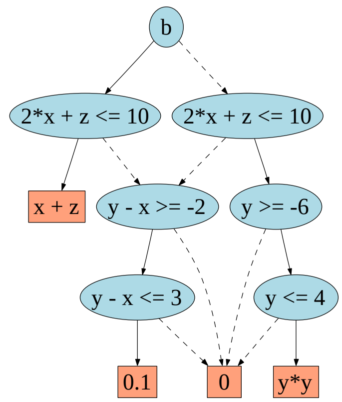


Figure 2: An XADD example.

ADD (XADD) formalism introduced in (Sanner, Delgado, and de Barros 2011) and demonstrated in Figure 2.

The XADD is like an algebraic decision diagram (ADD) (Bahar et al. 1993) except that (a) the decision nodes can have arbitrary inequalities (one per node) and (b) the leaf nodes can represent arbitrary functions. The decision nodes still have a fixed order from root to leaf and the standard ADD operations to build a canonical ADD (REDUCE) and to perform a binary operation on two ADDs (APPLY) still apply with minor modifications in the case of XADDs.

Of particular importance is that the XADD is a directed acyclic graph (DAG), and hence is often much more compact than a tree representation, e.g., as demonstrated in Figure 2. Furthermore, one can use the feasibility checker of an LP solver to incrementally prune unreachable branches in the XADD DAG — an operation crucial for maintaining XADD compactness and minimality (Sanner, Delgado, and de Barros 2011). We remark that not only are XADDs compact on account of their reconvergent DAG structure (each path from root to leaf would be a separate case partition), but that all unary and binary operations on XADDs can directly exploit this compact DAG structure for efficiency.

All XADD operations except for definite integration have been defined previously; we note that unlike ADDs, some XADD operations like $\max(\cdot, \cdot)$ and $\min(\cdot, \cdot)$ can introduce out-of-order decisions which can be easily detected and repaired as discussed in (Sanner, Delgado, and de Barros 2011). Extending the definite integration operation to XADDs is straightforward: treating each XADD path from root to leaf node as a single case partition with conjunctive constraints, $\int_{v=-\infty}^{\infty}$ is performed at each leaf and the result accumulated via the $\oplus$ operation to compute (6).

Computational Complexity

For a graphical model inference algorithm like SVE, it is a natural question to wonder if space and computation time can be bounded in terms of the tree-width of the underlying graph, as for purely discrete models. The short answer is no. While a factor over many variables may be represented compactly as a piecewise expression (unlike, e.g., a tabular enumeration in the discrete case), one can generally only upper bound the number of pieces needed in a case expression (and hence computation time and space) as an exponential function of the number of primitive binary operations ($\oplus, \otimes, \max, \min$) used by SVE — assuming the PPFG has some factor with at least two non-zero case partitions. Since one definite integral during SVE can easily require 10's or 100's of primitive case operations, one can see that either SVE will be intractable *or* that these worst-case upper bounds are extremely loose when using data structures like the XADD. Fortunately, the latter proves to be the case as shown next in the empirical results.

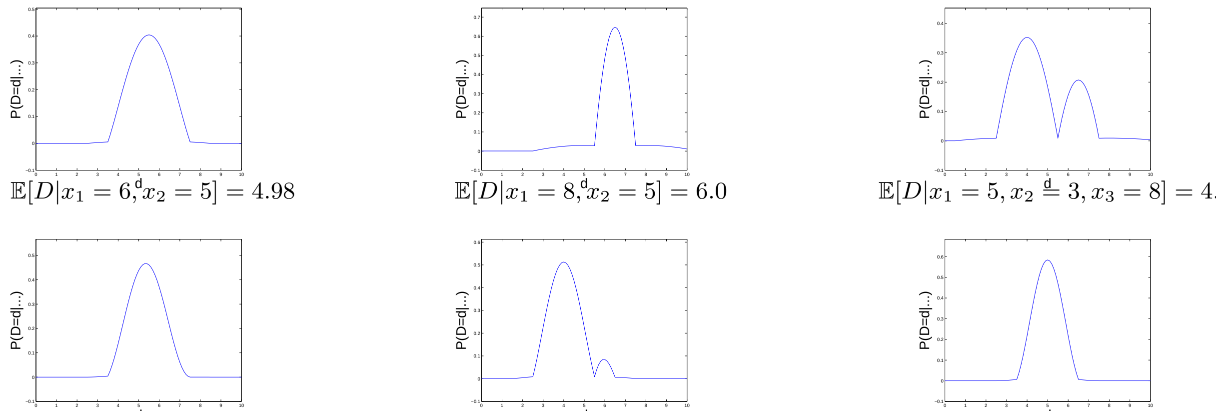
Empirical Results

In this section we present proof-of-concept experiments for the robot localization graphical model provided in the introduction and a basic discrete/continuous tracking task in Figure 4. Our objectives in this section are to show that the previously defined exact inference methods can work with reasonably sized graphical models having multiple continuous and discrete variables with complex piecewise polynomial distributions and that SVE can exactly compute the highly complex, multi-modal queries in an exact functional form — a milestone in exact closed-form inference for graphical models with such complex combinations of distributions.

Localization: For the robot localization task, we plot the posterior over the distance $d \in D$ given various observations in Figure 3. What is noteworthy here is the interesting multi-modal nature of these plots. Recalling the discussion in the introduction, the range finder was modeled with various error modes; because of these models, the posterior plots demonstrate the different combinatorial possibilities that each measurement was accurate or noise with the various peaks in the distribution weighted according to these probabilities. Despite the multi-modal nature of these distributions, the conditional expectation of D can be computed exactly and is shown below each posterior.

We note that in a Bayesian risk setting, where these posterior beliefs may be multiplied by some position-based risk function (e.g., one that increases with proximity to stairwells), it is important to have this true multi-modal posterior rather than an inaccurate unimodal approximation. And while sampling may work well for fixed expected risk calculations, if the evidence is changing or one wants to perform sensitivity analysis of risk subject to differing evidence (i.e., conditioning on unassigned evidence), one must collect samples for each specific evidence case under consideration.

Tracking: The tracking graphical model and (conditional) distributions are shown in Figure 4 and works as follows: an object has hidden state $x_i \in \mathbb{R}$ defining it's x position that is sampled for time steps $i = 1 \ldots t$. On every time step a noisy observation $o_i \in \mathbb{R}$ is received regarding the position of the object. Also at every time step, a variable b_i indicates whether the sensor is broken (1) or not (0). A sensor fails with probability 0.2 and stays broken once it breaks. If the sensor is not broken, the observation model for o_i is a symmetrical triangular distribution centered on the underlying state x_i; if it is broken the observation model is an asymmetric triangular distribution with a peak at the true x_i state, but biased to underestimate. The initial position x_1 is



$\mathbb{E}[D|x_1 = 6, x_2 = 5] = 4.98$ $\mathbb{E}[D|x_1 = 8, x_2 = 5] = 6.0$ $\mathbb{E}[D|x_1 = 5, x_2 = 3, x_3 = 8] = 4.39$

$\mathbb{E}[D|x_1 = 5, x_2 = 5, x_3 = 6] = 4.98$ $\mathbb{E}[D|x_2 = 1, x_2 = 3, x_3 = 4, x_4 = 8] = 5.45$ $\mathbb{E}[D|x_1 = 5, x_2 = 4, x_3 = 6, x_4 = 5] = 4.89$

Figure 3: Queries for robot localization. The diagrams show $p(d|\mathbf{e})$ vs. d for the given evidence $\mathbf{e}$ shown below each diagram along with the exactly computed expectation $\mathbb{E}[D|\mathbf{e}]$ for this distribution.

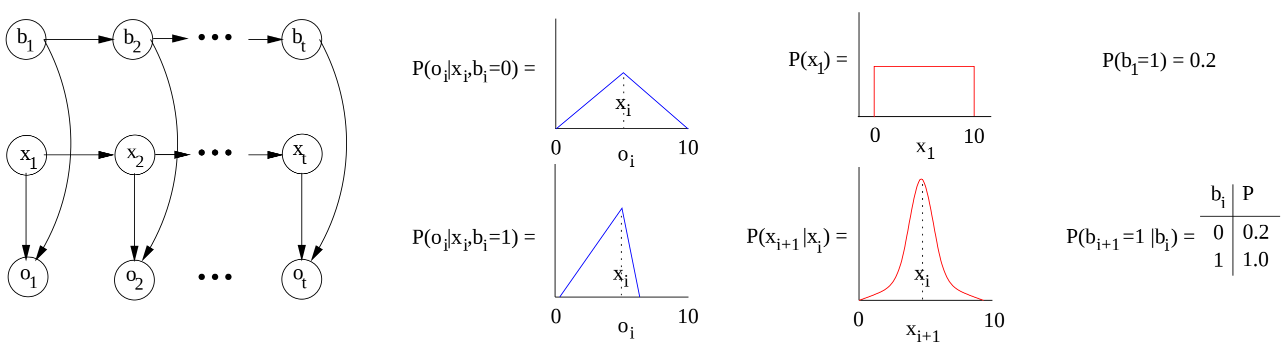
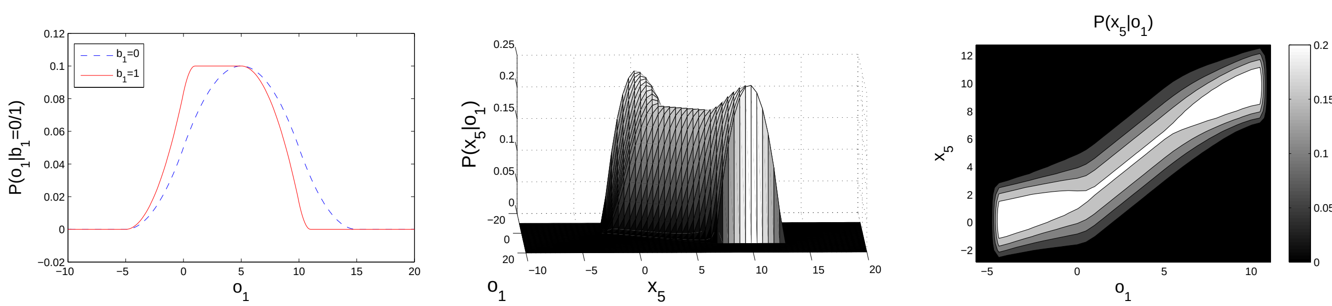


Figure 4: The tracking graphical model and all conditional probabilities.



(a) (b) (c)

Figure 5: Plots of query and evidence variables for three queries in *tracking*: (a) $p(o_1|b_1 = 0)$ (blue dash) and $p(o_1|b_1 = 1)$ (red solid). (b) $p(x_5|o_1)$ as a 3D plot, and (c) $p(x_5|o_1)$ as a contour plot.

uniformly distributed between 0 and 10 and each subsequent x_{i+1} is sampled from a truncated Gaussian centered on x_i.

In Figure 5, we show results for three queries in the tracking model. Result (a) demonstrates an asymmetric posterior while results (b) and (c) demonstrate the complex multimodal posterior distribution over x_3 conditioned on uninstantiated continuous evidence o_1 and the ability of the SVE inference algorithm to compute this in an exact closed-form. All of these queries completed in under 1 minute demonstrating that despite the complex symbolic manipulations involved and multidimensional/multimodal inference results, the SVE method can effectively compute closed-form exact inference for this non-Gaussian tracking task.

The important point to observe here in both Figures 3 and 5 is that using SVE we are able to derive various multimodal and highly irregular posterior distributions in an exact functional form. And due to our symbolic form, we are able to derive this distribution as a function of uninstantiated evidence as in Figures 5(b,c) — inference that is not easily done (if at all) via standard sampling or numerical integration techniques. Such symbolic forms are useful when we want to analyze arbitrary variable dependencies in complex systems with sophisticated distributions that can be specified (or arbitrarily approximated) as PPFGs.

Related Work

Almost all prior work on continuous variable graphical models — with the exception of Kalman filtering (Welch and Bishop 1995) and other jointly Gaussian models (Weiss and Freeman 1999) — has focused on *approximate* or *non-closed-form, exact* inference. Such methods for non-Gaussian continuous graphical model inference include

- *Discretization*: arbitrary accuracy but subject to the curse of dimensionality,
- *Numerical Integration*: arbitrary accuracy but subject to the curse of dimensionality, also only applies to instantiated continuous evidence — rather than uninstantiated evidence as shown for the SVE query in Figures 5(b,c),
- *(MCMC) Sampling*: exemplified in the expressive BUGS language and inference (Lunn et al. 2000), aymptotically unbiased but only applies to instantiated evidence and known to be slow to converge when probabilities are nearly deterministic (note that SVE is unaffected by this),
- *Projected Message Passing* such as variational inference (Jordan et al. 1999) and expectation propagation (Minka 2001) that can produce a functional form for a query result, but with an *a priori* assumed projective distribution (often Gaussian) that may not be appropriate in practice, cf. Figures 5(b,c), and
- *Mixtures of Truncated Exponentials (MTEs)*: an approach introduced in (Moral, Rumí, and Salmerón 2001) not unlike that taken here and further developed extensively in the approximate case – even with trees (Moral, Rumí, and Salmerón 2003; Rumí and Salmerón 2005; Cobb, Shenoy, and Rumí 2006), but only for non-oblique pieces (hyperrectangular piece boundaries) that may require arbitrary space to accurately approximate functions that can be exactly represented with oblique pieces as in Figures 5(b,c).

In short, it seems that little progress has been made in *closed-form exact* inference of results in a functional form for *expressive* non-Gaussian joint distributions and uninstantiated continuous evidence — as done for the oblique piecewise polynomial distributions in this paper.

Concluding Remarks

SVE and its PPFG formalism using oblique piecewise polynomials represents a novel and expressive class of *non-Gaussian* factors for which integration is closed-form, thus enabling exact symbolic inference for any probability queries and expectations in this model. Most realistic distributions have bounded support and can be arbitrarily approximated by piecewise polynomials, hence this work opens up the possibility of exact solutions (or arbitrary approximations thereof) in discrete/continuous graphical models for which previous exact solutions were not available. As such, we believe that SVE provides a significant step forward and a new alternative for the difficult task of *exact closed-form* inference in discrete and continuous graphical models.

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